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Macroeconomic Model of Public Debt Servicing Capacity and Debt Management

Debt is not a matter of concern as long as it is manageable and sustainable. Debt management is the process by which the government acquires and uses the debt effectively and efficiently. Debt is manageable as long as the cost of acquiring debt is reasonably low and debt obtained is used efficiently in such a way that it helps growth and efficient allocation of resources in the long run. Debt is used efficiently if the ratios of debt service to total revenue and external debt service to exports fall or remain constant. The underlying assumption is that the projects for which borrowed money is used would generate sufficient output and exports for debt repayment.

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I: Introduction

Debt is not a matter of concern as long as it is manageable and sustainable. Debt management is the process by which the government acquires and uses the debt effectively and efficiently. Debt is manageable as long as the cost of acquiring debt is reasonably low and debt obtained is used efficiently in such a way that it helps growth and efficient allocation of resources in the long run. Debt is used efficiently if the ratios of debt service to total revenue and external debt service to exports fall or remain constant. The underlying assumption is that the projects for which borrowed money is used would generate sufficient output and exports for debt repayment.

Public debt management typically involves activities ranging from the formulation of a debt/borrowing strategy. This strategy is generally based on country's debt situation, financial market situation, need of new finances, use of borrowed money, meeting of debt service obligations on time and maintenance of information systems and databases.[1] These activities need to be governed under an explicit and clear legal mandate and organized under a framework where roles and responsibilities of the agencies involved are well specified. The objective of this study is to present some methods to evaluate the debt servicing capacity of a country. In this context, an effort is made to present a macroeconomic framework for debt retirement of a country. The study is organized as follows. In section II, we present the method to evaluate the debt servicing capacity of the country. Section III presents the debt retirement simple model with no service payments. In section IV, we present the model of debt retirement in case of interest payments. Section V presents the summary and conclusion.

[1] Often referred to as the front, middle and back office functions; for more detail see "Guidelines for Public Debt Management," World Bank and International Monetary Fund (February 2001).

II: Methods to Evaluate the Debt-Servicing Capacity of a Country

High debt leads generally to high debt service liability. However, the severity of the debt service liability of a country depends on the relationship of its debt to its GDP and on the level of debt in relation to its debt service obligations. As such, the absolute volume of debt of a country is not perhaps as much a matter of concern as the extent of debt service liability. The level of debt service liability of a country may be high or low depending on its level of economic development. The incidence of debt service liability of a borrowing country is generally reflected from debt indicators expressed as ratios of debt to GNP (Debt-GNP), debt to debt-service liability (Debt-Service), debt-service to foreign exchange earnings (Debt-FEE), etc. The debt service payment may be large or small depending on the values of such debt indicators and the conditions under which loans are sought and received. As such, the values of these ratios and thereby the severity of debt burden keep changing in response to changes in the terms of borrowing and overall economic conditions of a country.[2]

Debt service liability is assessed by methods of percentages. Briefly, debt and debt-service charges are expressed as percentages of annual values, for examples, of GNP, export, foreign exchange earnings, imports etc in different ways. The methodology of assessing the long-run debt-servicing capacity of a country is based on the comparison of costs and benefits of public debt used in the process of economic development. There are two methods of determining the debt-service capacity of a country. First method commonly used to compare costs and benefits of public debt is the critical interest rate (CIR) approach. The second approach is the identification of the limit value of the Debt-GDP ratio. The CIR indicates the level of interest rate that makes the growth rate of public debt equal to the growth rate of GDP. It is also the maximum interest rate that can be paid on loans while maintaining at the same time a desirable debt-GDP ratio. In principle, if the average interest rate on loans exceeds the CIR,

[2] Avramovic et al. (1964), Aliber (1980), Nowzad & William (1981) and Lee (1983) very well highlighted the desirability and appropriateness of such indicators.

debt will increase faster than GDP leading thereby to an ever-increasing debt burden. Algebraically, the CIR is calculated as:

$$CIR = \frac{g(s_1 - s_0)}{(kg - s_0)} \quad (1)$$

where, g is growth rate of GDP, s_1 marginal saving rate, s_0 average saving rate at the beginning of the period, and k is incremental capital-output ratio. For the purpose of calculation, the values of *CIR,* g and k can be calculated for different time periods to know how the debt-servicing capacity of a country may have changed.

The optimal borrowing capacity can also be found conceptually by using a simpler model.[3] Technically, optimal external borrowing is a function of its costs and benefits. The terms at which public debt can be obtained are crucial in determining the cost-benefit ratios. The objective is to obtain loans at such an interest rate as renders debt/GDP ratio stable over time. A stable debt/GDP ratio may depend on a particular relationship among relevant variables. The equation mentioned below connects the required variables in a relationship, which determines the convergence of external debt to a stable ratio in terms of GDP.[4]

$$\frac{D}{Y} = \frac{(kg - s)}{(g - i)} \quad (2)$$

Where, D, Y, k, g s and i denote public debt, Gross Domestic Product GDP, incremental capital output ratio, growth rate of GDP, marginal saving rate and interest rate on public debt. The equation clearly shows that debt-GDP ratio is an increasing function of the interest rate on public debt. To determine the effect of a change in GDP growth rate on the debt-GDP ratio, consider the following derivative:

[3] Avramovis (1964), Solomon (1977), Nowzad (1981) and Lee (1983) highlighted the use of simple model in their studies.
[4] For detail see Lee (1983).

$$\frac{\partial \left(D/Y \right)}{\partial g} = \frac{(g-i)k - (kg-s)}{(g-i)^2} = \frac{s-ik}{(g-i)^2}$$

$$\frac{(s-ik)}{(g-i)^2} > 0 \quad \text{if} \quad i < \frac{s}{k} \quad \text{and} \quad \frac{(s-ik)}{(g-i)^2} < 0 \quad \text{if} \quad i > \frac{s}{k}$$

It follows that debt-GDP ratio will increase or decrease with the growth rate of GDP depending on whether the interest rate on public debt is less than or greater than the ratio of saving rate to incremental capital output ratio.

The values of different debt indicators and CIR basically provide a future guideline for negotiations of debt restructuring or future loan negotiations. Inept management of debt and regularly rising debt to GDP are likely to induce changes in the main macroeconomic indicators like crowding out of investment, fiscal instability, inflationary pressures and exchange rate fluctuations etc. Further, rising debt burden has also many undesirable implications for the country. There is, therefore, a need to make serious attempts at finding a sustainable indigenous solution of the debt. Since debt becomes a concern when it crosses manageable and sustainable limits, policy makers are suggested to formulate a process by which government is forced to use loans effectively. Debt is used efficiently as long as the level of external debt-service and the ratio of debt service to total revenue are either falling or at least remaining constant. Public debt can be sustainable as long the projects financed with borrowed money generate sufficient output and export earnings for debt repayment. Further, it is required to maintain capital output ratio and higher marginal saving rate to avoid unsustainable rate of interest on borrowed money.

III: Macroeconomic Model of Debt Reduction

This section presents a macroeconomic model of debt retirement based on Harrod-Domar two-gap model of economic growth. We extend Harrod-Domar growth model to allow the role of public debt in the process of economic growth. We introduce a terminal time in the model as a policy decision in order to analyze the trade-off between various objectives of development planning such as the trade-off between the desire to become independent of future borrowing and to achieve a high growth rate of GDP. Only after an assessment of this trade-off, it can be decided to undertake the task and policies consistent with the overall development objectives.

Presumably, the independence from future borrowing does not mean that the country will be free from all kinds of debt liabilities such as foreign private investment, funds for amortization and interest payments on loans. All that is meant by this independence from borrowing is that after some specific time there should be no increase in debt liability and the country should be free from the need of further borrowing. By that time domestic resources should exceed the investment required to ensure output growth equivalent at least to the service payments due from all forms of public debt which were obtained in the past.

To begin with, we discuss some general issues relevant to the question of independence from future borrowing and some quantitative assessment of its implications. The analysis is then extended to consider the implications of public debt on the debt-service obligations, which are amortization and interest payments on borrowed loans.

IV: The Simple Model with No Service Payment

National income identity shows that at any point of time, the total amount of resources available to a country which it can consume and set aside for (gross) investment or use for exports is equal to the country's gross national product plus its imports. It follows from national income identity that at any time the difference between a country's gross investment and gross domestic savings is necessarily equal to the difference between its imports and exports. Therefore a country can invest more than it can save, and thus achieve a growth rate higher than that determined by its domestic savings rate, if it can import more than its exports by the same amount. This will naturally depend upon how it finances the import surplus. Whether the country relies on private net capital inflow or borrows publicly, a net borrowing remains the only offsetting item in the balance of payments to finance the deficits. Thus, the role of public debt in a growing economy is quite obvious. A given amount of public debt can finance an equal amount of import-surplus of the capital receiving country and also allow its investment to exceed its domestic saving by the same amount, that is:

$$I_t - S_t = M_t - X_t = F_t \qquad (1)$$

Where I_t ,S_t ,M_t,,X_t and F_t denote gross investment, gross domestic savings, imports, exports and net funding (borrowing), respectively. If gross incremental capital-output ratio is denoted by k, and growth rate of GDP by g, then:

$$I_t = kgY_t \qquad (2)$$

Where Y_t is GNP at time t. The rate of savings is expected to rise over time. Let s_0 be the average rate of savings at the initial period, that is period zero, and s_m the incremental savings rate between period zero and t. We assume that the saving function is of the following form.

$$S_t = s_0 Y_0 + s_m (Y_t - Y_0) = (s_0 - s_m) Y_0 + s_m Y_t \qquad (3)$$

Substituting from (2) and (3), we can write (1) as follows:

$$kgY_t - \left[(s_0 - s_m)Y_0 + s_m Y_t\right] = F_t \qquad (4)$$

$$g = \frac{1}{k}\left[(s_0 - s_m)\frac{Y_0}{Y_t} + s_m + \frac{F_t}{Y_t}\right] \qquad (5)$$

Equation (5) gives us a unique relationship between the growth rate of GDP, g, and the net borrowing as a proportion to GDP i.e F_t/Y_t. Given the incremental capital-output ratio, marginal rate of savings, average savings rate at the initial period and the initial and the current level of GDP; we can obtain either the net borrowing required by a target rate of growth, or the rate of growth that can be achieved with a given amount of borrowing. Since $\partial g/\partial F > 0$, a large borrowing can achieve a higher rate of GDP growth, keeping all other factors constant.

The above formulation can also be used to measure the capacity of the country to absorb borrowed funds. The proposition on which the above argument i.e. the rate of growth can be raised by increasing the rate of investment financed by borrowing is based, implicitly assumes that capital, not labor, is the main bottleneck for raising the rate of growth.[5] This way of looking at the relationship between borrowed money and the rate of growth of GDP implies that every year the import–surplus somehow gets adjusted to the net inflow of public debt. Total imports adjust automatically to the two exogenous variables i.e. X_t and F_t and the growth rate entirely depends on how far F_t can eliminate the bottleneck of the low rate of domestic savings. But imports may not be so freely adjustable in an underdeveloped country and thus a more comprehensive model will inevitably contain additional constraints. For

[5] In case of an underdeveloped economy, this assumption may be quite unrealistic. Even if there is an unlimited supply of capital and labor, the rate of growth may have an upper bound determined by the supply of complementary factors like skilled labor, managers, entrepreneurs, etc. In that case there is an optimum rate of receipt of borrowed capital which is equal to the excess of rate of investment, required to realize the maximum possible growth rate determined by the other non-capital factors, over the rate of savings. This excess requirement basically will determine the absorptive capacity for borrowed funds. A large amount of debt beyond the absorptive capacity of a country will fail to raise the rate of growth (Harrod, 1963).

that purpose, we assume that there is a rigid relationship between the level of imports required and the level of GDP. The assumption that there is a minimum import requirement corresponding to a level of GDP can be interpreted in two ways:

First, the import requirement is in fixed proportions to consumption, gross investment and exports, which, as a first approximation may be regarded as the same. Such an assumption may be questionable for an underdeveloped economy at an early stage of development where imports to consumption ratio may be much higher due to the imports of some basic necessities for supporting its subsistence level of consumption. The case may be similar with exports if they are not completely given exogenously. But even if there exist consumption and export basket with negligible minimum import requirements, it is quite possible that gross investment required for maintaining that level of consumption and exports over time may need some minimum level of imports. The minimum level of imports is required simply because certain right types of equipment, machinery, etc. are not domestically produced.

In the case where only a given amount of loan is available to finance the gap between savings and investment, the funding may not achieve the target of growth rate of GDP as determined by the capital output ratio. This is because imports may increase due to increase in income and this increase in imports may exceed the exports by more than the given amount of borrowed funds.

In either case, the condition for a given amount of debt to achieve a rate of GDP growth determined by the amount of savings- investments gap it can finance, is that $M \leq X+F$, with M accounting for the minimum level of imports. If this condition is not fulfilled, there will be two operative constraints on the growth level of GDP, namely the resource constraint $I-S \leq F$

and the balance of payment constraint M-X ≤ F. The solution of the system will depend upon which of the two constraints is more restrictive, i.e. whichever gives the lower solution.[6]

It is also adopted that this upper bound on GDP growth rate, determined by the import constraint becomes less and less restrictive with time as rising GDP is likely to be accompanied by increasing availability of domestically produced capital goods. Thus, in the long run, the saving-investment gap that will determine the effect of borrowing capital on the growth rate of GDP, although in the early phase of development the import constraint may be more important.

The following analysis is mainly focused on the debt inflows filling the saving-investment gap. Although, this renders the analysis partial in nature, but it may be still worth while mainly for two reasons:

(a) Even if there were no import constraints, the effect of public debt on growth rate will depend on how far it overcomes the savings constraint; and

(b) It may be maintained that in the long run it is the saving constraint that is more restrictive than the imports constraints.

Given the structure of our model it is possible to calculate the number of periods after which a country will generate enough domestic savings to finance the investments required to attain a given GDP growth rate without depending on public borrowing. This is derived from equation (4) reproduced as follows:

$$(kg - s_m)Y_t + (s_m - s_0)Y_0 = F_{tt} \qquad (6)$$

In the nth year when $F_n = 0$, the equation (6) may be rewritten as:

Setting $F_n = 0$ in the terminal period and re-arranging (6), we obtain:

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[6] Chenery and Bruno (1962) introduced a variant of the first kind of import restriction mentioned above and estimated import requirements from the solution of an input-output model under some strict assumptions. They observed that the import bottleneck at lower levels of GNP is more restrictive than the savings bottleneck.

$$Y_n = \left[\frac{(s_m - s_0)}{(s_m - kg)} \right] Y_0 \quad (7)$$

Since $Y_n = Y_0 (1+g)^n$, we can further write this equation as:

$$(1+g)^n = \left[\frac{s_m - s_0}{s_m - kg} \right] \quad (8)$$

Equation (8) permits the determination of period (n) to get independence from public borrowing with a given target growth rate of GDP.[7]

It is obvious that the essential condition for a country to become independent from public debt at a future date is that $\lfloor s_m > k.g \rfloor$, that is, the marginal rate of savings should exceed the required rate of investment. Further, the value of (n) will be smaller, i.e. the year of self-sufficiency will be closer, the larger the difference between the marginal rate of savings and the required investment and the higher the value of the initial rate of saving (s_0).

Assuming that $g > 0$ and $s_m > s_0$, it must be emphasized that Equations (4), (5) and (8) refer all to the case of public debt taking place purely in the form of interest free borrowings. No interest payment obligation in attached to them, and after the nth year, that is, when the year of independence from public debt is reached, the country's domestic savings will exceed its required investment rate. In that case, if the same rate of growth is maintained, the country will be able to lend money instead of borrowing from different sources. Otherwise the country's rate of growth will have to increase every year so that the rate of investment

[7] This assumption will be relaxed in Section IV. From Equation (7) we can also calculate the number (n) of years needed for minimum-import requirements to be fully financed by the country's exports.
Then $M_n = X_n$ og $m_n\, k.\, g.\, Y_n = Xn$
Assuming a constant rate of growth over time in GDP and exports, we get
$m\, k.\, g.\, Y\,(\,1 + g\,) = X\,(\,1 + e\,)$
$(1+g)^n / (1+e)^n = [\,1/ m_n\, k.g]\, X_0 / Y_0$;
from which the value of n can be obtained. This n need not be equal to the n obtained from equation (8).

required to achieve higher rate of growth is equal to the increasing domestic savings. The model then will look like a simple Harrod-Domar system.

$$I_t = S_t \qquad (9)$$

Substituting for the saving and investment function from (1) and (2) and re-arranging, we obtain:

$$g = \left(\frac{1}{k}\right)s_m - \left(\frac{1}{k}\right)(s_m - s_0)\frac{Y_0}{Y_t} \qquad (10)$$

As time passes, this increasing rate of growth will approach $\left[s_m / k\right]$ as the expression $\left(\frac{1}{k}\right)(s_m - s_0)\frac{Y_0}{Y_t}$ will approach towards zero over time.

To illustrate how this method can be applied in practice, we assume that if country needs to get independence from borrowing at some specific time in future, say after 15 years between 2009-10 and 2024-25, i.e. no borrowing aid will be required by 2024-25.[8] Here we concentrate only on savings-investment gap, assuming implicitly that import, exports and other relevant variables get somehow or other correctly balanced.[9] Thus, if savings- investment gap is eliminated by the year 2024-25, then it is possible that country may, though not necessarily, will become independent of borrowing by that time. The other assumption of this exercise is that the savings-investment gap is covered by foreign assistance coming in the form of grants only with no repayment obligations. This is obviously unrealistic, but it is justified because this gives the minimum that will be required for attaining independence from debt in fifteen years.

[8] This exercise is performed only for the sake of illustration. We merely try to find out what are the consistent values of different magnitudes if this claim is borne out in practice.
[9] In the following analysis, we incorporate other gaps as well.

The following Table 1 gives on assuming GDP growth of around 7 percent per annum, the consistent values of gross marginal capital-output ratio and the marginal rate of savings, which will enable the country to become self-supporting after fifteen year (n= year of independence from foreign aid=15).[10]

TABLE 1

Different Values of Marginal Rate of Savings and Marginal Capital-Output Ratios to get independence in 15 years with given g= 7 % and average saving rate =10%.

K	S
2.5	21.5%
3.0	27.5%
3.5	32.4%
4.0	37.9%

To get independence from debt in 15 years, table 1 gives different values of the marginal rate of savings and marginal capital-output ratios, if the expected 7 percent rate of GDP growth and average saving rate of 10 % is taken as the target growth rate. This table shows that for the country to become independent by 2015 from debt, and to achieve the rate of growth of about 7 percent would require as low a marginal capital output ratio as 2.5 and an incremental savings rate of about 22 percent.

All this seems to be too ambitious, as the available empirical evidence would suggest that a marginal capital output ratio of 2.5 is too low or a marginal rate of savings of 22 percent is too high. In general, the figures of the table 1 indicate the amount of effort in terms of raising the rate of savings or of productivity of investment involved in any attempt to achieve independence from borrowing at an early date.

[10] Using formula $[1+g]^n = (s_{t-1} - s_{t-1}) / s_{t-1} - k.g$
which can be written as $s = [k.g (1+g)^n - s_{t-1}] / [1+g]^n - 1$
We can calculate marginal rate of savings for different growth rates and different capital-output ratios given average savings rate and number of years to get independence from debt. Similarly we can calculate the required capital-output ratios for given marginal, average savings rate and GDP growth rate. This analysis is more valid if we incorporate the amortization period and debt-service obligations in our analysis. We complete this analysis in the following text where we incorporate amortization and debt-service obligations.

We can make use of Equation (9) and find the year of independence from interest free loans taking into account different values of GDP growth rates and marginal saving rate. This sensitivity analysis is extended to incorporate different values of capital output ratio as well. The results of our exercise are presented in Table 2, 3 and 4.

The results show that the number of years required to achieve such independence with a stipulated rate of growth is highly sensitive to variation in marginal capital-output ratio and marginal rate of savings. The analysis shows that the reduction in the years of dependence is more responsive to a decrease in k than to an increase in the value of s, i.e. the system is more sensitive to variation in incremental capita-output ratio (k) than in incremental saving rate(s). If k rises by 20 percent from 2.5 to 3.0, the value of s must rise by more than 25 percent i.e. from 21.5 percent to 27 percent to achieve independence in the given fifteen years. Table 2 and table 3 show the results obtained by using different values of capital-output ratio, marginal saving rates, GDP growth rate. We use average saving rate at 10 percent, in table 3, in order to find the variation in the period to get independence from borrowing when average saving rate is low.

Table 2

Different Values of the Year of Independence from Foreign Borrowing for Different Values of GDP Growth Rate and Marginal Rate of Savings (given k=2.5,3 & 3.5 and average saving rate=12%)

g / s	K=2.5			K=3			K=3.5		
	0.04	0.05	0.06	0.04	0.05	0.06	0.04	0.05	0.06
0.15	13.02	3.73					28.01		
0.16	10.33	2.73	23.79		28.41		17.67		
0.17	8.57	2.15	15.72		18.78		13.02		
0.18	7.33	1.78	11.89		14.20		10.33	50.93	
0.19	6.40	1.51	9.60		11.46	33.39	8.57	31.57	
0.20	5.68	1.32	8.06		9.63	23.79	7.33	23.83	

Note: Blank cells in the table show that the necessary condition $\left[s_m > k.g\right]$ to get independence from foreign borrowings is not fulfilled.

Table 3

Different Values of the Year of Independence from Foreign Borrowing for Different Values of GDP Growth Rate and Marginal Rate of Savings (given k=2.5,3 & 3.5 and average saving rate=10%)

g / s	K=2.5			K=3			K=3.5		
	0.04	0.05	0.06	0.04	0.05	0.06	0.04	0.05	0.06
0.15		14.20		13.02			41.03		
0.16		11.04	30.74	10.33	36.72		28.01		
0.17		9.05	21.49	8.57	25.67		21.60		
0.18		7.67	16.83	7.33	20.10		17.67	56.82	
0.19		6.66	13.91	6.40	16.62	37.70	14.98	36.72	
0.20		5.89	11.89	5.68	14.20	27.62	13.02	28.41	

Note: Blank cells in the table show that the necessary condition $\lfloor s_m > k.g \rfloor$ to get independence from foreign borrowings is not fulfilled.

The results show relative sensitivity of the system to variations in the marginal capital output ratio and marginal savings rate once we assume a given level of GDP growth rate and initial level of domestic saving rate. This is of some interest in relation to the question: how much additional effort is required in the form of raising the marginal rate of savings if the marginal capital- output ratio turns out to be higher than expected so that the target year of independence is actually realized.[11] On the basis of our results in table 2 and table 3, we can say if the number of years to reach independence from debt (without interest payment) is a given objective, and if the marginal capital-output ratio exceeds the projected value by x percent, then the marginal rate of savings must be raised over its projected value by more than *x* percent so that the given objective is possible to realize.

[11] Marginal capital-output ratio is a technological variable which is responsive to policy variations.

We can prove this claim mathematically as:

Considering Equation (8):

$$(1+g)^n = \left[\frac{s_m - s_0}{s_m - kg}\right]$$, We can further re-write as:

$$n = \frac{1}{c}\left[\ln(s_m - s_0) - \ln(s_m - k.g)\right] \qquad (11)$$

where, c = log (1+g), a constant for a given rate of growth g. Since s_m, k, and s_0 are mutually independent, taking partial derivatives with respect to s_m , we write

$$\frac{\partial n}{\partial s_m} = \frac{1}{c}\left[\left\{\frac{1}{s_m - s_0}\right\} - \left\{\frac{1}{s_m - k.g}\right\}\right] = -\frac{1}{c}\left[\frac{k.g - s_0}{(s_m - s_0)(s_0 - k.g)}\right] \qquad (12)$$

Since $s_m > k\,g > s_0$, by assumption, this shows that a rise in s_m will reduce (n), the numbers of years taken to reach independence.

Similarly, $$\frac{\partial n}{\partial k} = \frac{1}{c}\left[\frac{g}{s_m - k.g}\right] \qquad (13)$$

which shows that a rise in k will raise n again assuming that ($s_m > k\,g > s_0$) .
In order that the effect of a rise in k is compensated by a rise in s_m or n, we get

$$\left(\frac{\partial n}{\partial s_m}\right) ds_m + \left(\frac{\partial n}{\partial k}\right) dk = 0 \qquad (14)$$

Substituting (12) and (13) into (14), we obtain:

$$\frac{1}{c}\left[\frac{(k.g - s_0)}{(s_m - s_0)(s_m - k.g)}\right] ds_m = \frac{1}{c}\left[\frac{g}{s_m - k.g}\right] dk$$

$$\left[\frac{ds_m / s_m}{dk / k}\right] = \left[\frac{g(s_m - s_0^t)}{k.g - s_0}\right]\frac{k}{s_m} > 1 \text{ since } s_m > kg > s_0$$

This proves the proposition that any increase in k will have to be compensated by a proportionally larger increase in s_m, so that the target of the year of independence is realized.

IV: The Model with Interest Payment

Until now, in our analysis, no reference has been made to the problem of interest payment on borrowing. Unless borrowing flows purely in the form of interest free loans, funds will have to be spared every year for interest payments in addition to and amortization payments. So there will be an annual drain equal to the payment of interest and amortization from the country's domestic savings. This drain in domestic savings has to be explicitly taken into account in the equation of borrowing requirements. The exact time profile of such re-payments depends upon the arrangements for amortization and interest payments for different kinds of loans. If both interest and amortization are charged on the net debt outstanding then loans are not completely liquidated but the unliquidated balance of each loan becomes smaller and smaller every year. Alternatively interest may be charged on the net debt outstanding, while the yearly amortization is fixed in equal installments of the original value of each loan. Yet another possibility is to fix equal installment method whereby amortization and interest on each loan are paid off in a series of equal installments.

There may be several other ways of charging interest and amortization on different kinds of loans subject to different methods of charging interest and amortization rates in any particular year. Quite often, some kind of grace period is attached to official loans so that the payments of interest and/or amortization do not start till some years after the loans are granted.

We further assume that all loans are given for a period of T years and the rate of interest "r" overtime is same for all loans. However, the estimation of debt requirements in this case is quite complicated. We consider explicitly the fact that after every *(T+j)* year, the *jth* year's loan is completely liquidated and no more repayment obligation is involved for it.

In case of public debt servicing requirements, the relationship between investment-saving gap and the inflow of foreign borrowing, as given by equation (1), need to be generalized.

Denoting the stock of public debt at the end of period t by D_t and the rate of interest on public debt by r_t, the time path of net foreign borrowing can be written as follows.

$$F_t = r_t D_{t-1} + \left(I_t - S_t\right) \quad (15)$$

In order to obtain a parameterized solution for net borrowing, we assume that the stock of debt in the initial period (period 0) is zero, the rate of interest is constant and the investment and saving functions are given by equations (2) and (3) respectively. Under these assumptions equation (15) can be solved for the equilibrium value of the public debt borrowing. It is shown in Appendix that such a solution is given by:

For $r \neq g$

$$F_t = \left(kg - s_m\right)(1+g)Y_0 \frac{r(1+r)^{t-1} - g(1+g)^{t-1}}{r-g} + \left(s_m - s_0\right) Y_0 (1+r)^{t-1} \quad (16)$$

For $r = g$

$$F_t = \left(kg - s_m\right) Y_0 (1+g\,t)(1+g)^{t-1} + \left(s_m - s_0\right) Y_0 (1+g)^{t-1} \quad (17)$$

The above equations can be used to calculate the number of years that are required to achieve independence from borrowing. Let the year when borrowing becomes zero be denoted by n. Then considering first the more general case of $r \neq g$, that is equation (16), the condition that $F_n = 0$ can be solved as follows.

$$\left(s_m - s_0\right)(1+r)^{n-1} = \left(s_m - kg\right)(1+g)\frac{r(1+r)^{n-1} - g(1+g)^{n-1}}{r-g} \quad (18)$$

This equation can be rearranged to solve for n as follows.

$$\left[\frac{s_m - s_0}{1+g} - r\frac{s_m - kg}{r-g}\right](1+r)^{n-1} = -g\frac{s_m - kg}{r-g}(1+g)^{n-1} \qquad (19)$$

Or

$$\left(\frac{1+r}{1+g}\right)^{n-1} = \frac{g\dfrac{s_m - kg}{r-g}}{r\dfrac{s_m - kg}{r-g} - \dfrac{s_m - s_0}{1+g}} \qquad (20)$$

Taking natural logs on both sides and solving for n yields:

$$(n-1)\log\left(\frac{1+r}{1+g}\right) = \log\left(\frac{g\dfrac{s_m - kg}{r-g}}{r\dfrac{s_m - kg}{r-g} - \dfrac{s_m - s_0}{1+g}}\right) \qquad (21)$$

$$n-1 = \frac{\log\left(\dfrac{g\dfrac{s_m - kg}{r-g}}{r\dfrac{s_m - kg}{r-g} - \dfrac{s_m - s_0}{1+g}}\right)}{\log\left(\dfrac{1+r}{1+g}\right)} \qquad (22)$$

The above equation solves the number of years by which borrowing becomes zero, in terms of all the other parameters of the system. In order to determine whether such a condition will exist in a particular situation, we consider the three possibilities: $r > g$, $r < g$ and $r = g$ one by one.

CASE A: R > G

When r > g, we shall have $\log\left(\frac{1+r}{1+g}\right) > 0$. Therefore for a meaningful solution for n (that is n > 1), the numerator in equation (6.22) must also be positive and hence we must have

$$\begin{aligned} & g\frac{s_m - kg}{r-g} > r\frac{s_m - kg}{r-g} - \frac{s_m - s_0}{1+g} > 0 \Rightarrow g\frac{1+g}{r-g} > r\frac{1+g}{r-g} - \frac{s_m - s_0}{s_m - kg} > 0 \\ & \Rightarrow g\frac{1+g}{r-g} - r\frac{1+g}{r-g} > -\frac{s_m - s_0}{s_m - kg} > -r\frac{1+g}{r-g} \Rightarrow 1+g < \frac{s_m - s_0}{s_m - kg} < r\frac{r(1+g)}{r-g} \end{aligned} \tag{23}$$

Or

$$\begin{aligned} & g\frac{s_m - kg}{r-g} < r\frac{s_m - kg}{r-g} - \frac{s_m - s_0}{1+g} < 0 \Rightarrow g\frac{1+g}{r-g} > r\frac{1+g}{r-g} - \frac{s_m - s_0}{s_m - kg} > 0 \\ & \Rightarrow g\frac{1+g}{r-g} - r\frac{1+g}{r-g} > -\frac{s_m - s_0}{s_m - kg} > -r\frac{1+g}{r-g} \Rightarrow 1+g < \frac{s_m - s_0}{s_m - kg} < r\frac{1+g}{r-g} \end{aligned} . \tag{24}$$

Thus it follows that with r > g, a country can attain independence from borrowing (its borrowing becomes zero) in a finite time period if:

$$\frac{r(1+g)}{r-g} > \frac{s_m - s_0}{s_m - kg} > 1+g \tag{25}$$

CASE B: R < G

With r < g we have $\log\left(\frac{1+r}{1+g}\right) < 0$. Therefore for n > 1, the numerator in equation (22) must be negative and hence we must have

$$r\frac{s_m - kg}{r-g} - \frac{s_m - s_0}{1+g} > g\frac{s_m - kg}{r-g} > 0 \Rightarrow r\frac{1+g}{r-g} - \frac{s_m - s_0}{s_m - kg} > g\frac{1+g}{r-g} > 0$$
$$\Rightarrow -\frac{s_m - s_0}{s_m - kg} > g\frac{1+g}{r-g} - r\frac{1+g}{r-g} > -r\frac{1+g}{r-g} \Rightarrow \frac{s_m - s_0}{s_m - kg} < 1+g < r\frac{1+g}{r-g} \quad (26)$$

Or

$$r\frac{s_m - kg}{r-g} - \frac{s_m - s_0}{1+g} < g\frac{s_m - kg}{r-g} < 0 \Rightarrow r\frac{1+g}{r-g} - \frac{s_m - s_0}{s_m - kg} > g\frac{1+g}{r-g} > 0$$
$$\Rightarrow -\frac{s_m - s_0}{s_m - kg} > g\frac{1+g}{r-g} - r\frac{1+g}{r-g} > -r\frac{1+g}{r-g} \Rightarrow \frac{s_m - s_0}{s_m - kg} < 1+g < r\frac{1+g}{r-g} \quad (27)$$

It follows that with r < g, a country can attain independence from borrowing in a finite time period if:

$$\frac{s_m - s_0}{s_m - kg} < 1+g < r\frac{1+g}{r-g} \quad (28)$$

CASE C: R = G

In the borderline case r = g the solution (6.18) is operative and the condition that $F_n = 0$ can be solved as follows.

$$(kg - s_m)Y_0(1+g\,n)(1+r)^{n-1} = -(s_m - s_0)Y_0(1+g)^{n-1} \quad (29)$$

Further simplifying and rearranging, we end up with the condition:

$$\frac{s_m - s_0}{s_m - kg} = 1 + g\,n > 1+g \quad (30)$$

To illustrate the effects of introducing repayment obligations into our model, we calculate the years of independence (repayment period of loans) with different stipulated rates of growth and uniform rates of service payments on the assumption of no outstanding debt at the initial period and the values of the marginal rate of savings equal to 15 percent, the initial average savings rate equal to 10 percent, and the marginal capital-output ratio equal to 3.5 and 3. Using equation (15) and (22), the year of independence when Fn=0 can be calculated.

The Table 4 shows that the number of years to get independence from public debt increases as growth rate and service payment rate increases. For example with low growth rate of 3 percent and service payment of just 3 percent, the year to get independence is almost 14 months. However higher growth rate of around 6 percent and service payment of 6 percent require almost 10 years to get independence from debt.

Table 4

Different Values of Repayment Time of External Debt for Different Values of "r" and "g" (given capital-output ratio= 3.5; marginal saving rate = 15 % and average saving rate=10%)

a↓ g→	0.03	0.04	0.05
0.03	1.2	5.4	4.2
0.04	1.5	6.4	6.5
0.05	2.1	7.7	9.6
0.06	2.6	9.0	10.4

These results testify our previous analysis of debt payment capacity that if we need to pay an interest rate of higher than critical interest rate then the debt is not sustainable. Table 5 is an extension of the previous table 4 with the change in capital output ratio from 3.5 to 3.

Table 5

Different Values of Repayment Time of External Debt for Different Values of "r" and "g" (given capital-output ratio= 3; marginal saving rate = 15 % and average saving rate=10%)

r↓ g→	0.03	0.04	0.05
0.03	1.8	6.4	9.0
0.04	2.3	7.2	9.5
0.05	3.2	8.4	10.0
0.06	3.8	9.1	10.5

The results in Table 5 and table 6 show that with lower capital-output ratio, the number of years to get independence increases and capital-output ratio is an important determinant of getting independence from public debt. With high growth rate and high service payment, the number of years to get independence from debt aid increases given capital-output ratio, marginal and average saving rate for different values of GDP growth rate and service payment ratio. The Table 4 shows the relative sensitivity of the system to variations in the marginal capital output ratio.

The relative sensitivity of the system to variations in the ambitious growth rate with high service payment is also found using some higher growth targets with high service payments and results are shown in Table 6 below.

Table 6

Different Values of Repayment Time of External Debt for Different Values of "r" and "g" (given capital-output ratio= 3; marginal saving rate = 25 % and average saving rate=10%)

r↓ g→	0.05	0.06	0.07	0.08
0.03	9.0	15.4	24.2	47.5
0.04	9.5	16.4	26.5	54.2
0.05	10.0	17.7	29.6	64.0
0.06	10.5	19.0	33.4	77.2
0.07	11.0	20.8	39.3	105.2
0.08	11.6	23.3	48.1	175.0

Nevertheless this exercise clearly shows the implication of having some repayment obligations associated with public debt of a country. If there were no repayment obligations in the initial period, then the country with a marginal rate of savings equal to 25 percent, the initial average saving rate of 10 percent, and a marginal capital output ratio of 3, would pay back all debt outstanding after 8.3 years if the growth rate were 5 percent.

The existence of some outstanding debt in the beginning will only postpone further the year of independence from public debt. According to figures in table 3, when loan required for service payment is 4 percent, it will take at least 9.5 or 16.4 years to pay back all debt if the target growth rate is 5 percent per year. If the target growth rate is 7 percent, it will take 27 years with the required rate of service payment of 4 percent, and 33.4 years with a required service rate of 6 percent. If the target income growth rate is 8 percent, it will take 54 years to pay back all loans with a required servicing rate of only 4 percent, and 105 years with a required rate of servicing of 7 percent. This means that when we require more loans for service payment and for increased growth rate, then the period to pay back loans also increases. The figures (table 5 and table 6) show that the system is highly sensitive to rate of GDP growth in relation to marginal rate of savings. As the rate of growth increases, the year of payment of loans also increases. When “r” is taken as the rate of service payments on loans, its low value indicates a prolonged period of amortization and low rate of interest.

In reality, overall debt comes to countries in the form of grants, loans and direct investment. This study has provided some methods of calculating the year of independence from public debt under some projected or assumed values of some relevant parameters. We have explained how difficult it is to become independent of debt, even under condition of requiring no fresh inflow of loans.

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APPENDIX

DERIVATION OF EQUATION (15)

The time path of net borrowing is given by.

$$F_t = r_t D_{t-1} + \left(I_t - S_t\right) \tag{A1}$$

In order to obtain a solution for F_t, we first need to trace the solution for the stock of foreign debt D_t. Since borrowing in period t is equal to the change in debt in period t over period t-1, the time path of the stock of debt can be expressed as a difference equation:

$$D_t = D_{t-1} + F_t \tag{A2}$$

Substituting for F_t from equation (A1), we obtain:

$$D_t = \left(1 + r_t\right) D_{t-1} + \left(I_t - S_t\right) \tag{A3}$$

Substituting backward, using the initial value $D_0 = 0$ and assuming that the rate of interest is constant, we obtain the following reduced form equation for the above difference equation:

$$D_t = \sum_{j=0}^{t-1} (1 + r)^j \left(I_{t-j} - S_{t-j}\right) \tag{A4}$$

Now using the investment and saving functions given by (2) and (3) the investment-saving gap in period t can be written as:

$$I_t - S_t = kgY_t - \left[s_0 Y_0 + s_m \left(Y_t - Y_0\right)\right] = \left(kg - s_m\right)Y_t + \left(s_m - s_0\right)Y_0$$
$$= \left(kg - s_m\right)Y_0 (1+g)^t + \left(s_m - s_0\right)Y_0 \tag{A5}$$

Now substituting from (A5), the time path of public debt given by (A4) can be expressed as follows.

$$D_t = \left(kg - s_m\right)Y_0 (1+g)^t \sum_{j=0}^{t-1} \left(\frac{1+r}{1+g}\right)^j + \left(s_m - s_0\right)Y_0 \sum_{j=0}^{t-1} (1+r)^j \tag{A6}$$

Using the solution for geometric series, we obtain:

$$D_t = \left(kg - s_m\right)Y_0 (1+g)\frac{(1+r)^t - (1+g)^t}{r-g} + \left(s_m - s_0\right)Y_0 \frac{(1+r)^t - 1}{r}, \quad r \neq g$$
$$= \left(kg - s_m\right)Y_0 (1+g)^t t + \left(s_m - s_0\right)Y_0 \frac{(1+g)^t - 1}{g}, \quad r = g \tag{A7}$$

To obtain the solution for the net borrowing, we substitute from (A5) and (A7) into (A1) to yield

$$F_t = r\left[\left(kg - s_m\right)Y_0 (1+g)\frac{(1+r)^{t-1} - (1+g)^{t-1}}{r-g} + \left(s_m - s_0\right)Y_0 \frac{(1+r)^{t-1} - 1}{r}\right]$$
$$+ \left(kg - s_m\right)Y_0 (1+g)^t + \left(s_m - s_0\right)Y_0 \quad r \neq g$$
$$= r\left[\left(kg - s_m\right)Y_0 (1+g)^{t-1}(t-1) + \left(s_m - s_0\right)Y_0 \frac{(1+g)^{t-1} - 1}{g}\right]$$
$$+ \left(kg - s_m\right)Y_0 (1+g)^t + \left(s_m - s_0\right)Y_0 \quad r = g \tag{A8}$$

Finally collecting common factors and further simplifying the expressions, we obtain the following solution for the net borrowing.

$$F_t = (kg - s_m)(1+g)Y_0 \frac{r(1+r)^{t-1} - g(1+g)^{t-1}}{r-g} + (s_m - s_0) Y_0 (1+r)^{t-1}, \quad r \neq g$$
$$= (kg - s_m) Y_0 (1+g\,t)(1+g)^{t-1} + (s_m - s_0) Y_0 (1+g)^{t-1} \qquad r = g \tag{A9}$$